Theoret. Chim. Acta (Berl.) 47, 147–156 (1978)

THEORETICA CHIMICA ACTA

On the Application of a Classical Perturbation Theory to the Theory of Coupled Fields

Hermann Hartmann and Kyu-Myung Chung

Institut für Physikalische Chemie der Johann-Wolfgang-Goethe Universität, Robert-Mayer-Str. 11, 6000 Frankfurt (Main), Federal Republic of Germany

A perturbation theory of coupled classical fields is developed on the basis of Hamilton's principle. In our investigation an individual description of an interacting many-particle system has been transferred, with the aid of a hydrodynamical model, to an anonymous field-theoretical description. The quantization of the fields has also been carried through and turned out to be simple. The application of the theory to liquid helium He^4 shows a good quantitative agreement with experiment. This is a consequence of our theory offering the possibility to take coupling between free and perturbed fields into account.

Key words: Coupled fields – Classical perturbation theory

1. Introduction

A classical perturbation theory for a system having one degree of freedom, and its generalization to a multidimensional system with a finite number of degrees of freedom was already developed earlier [1–3] on the basis of Hamilton's principle.

In the present paper, the extension of the theory to a system having an infinite number of degrees of freedom is reported on. In other words, classical fields interacting with each other can be treated perturbation theoretically.

The difficulty connected with a denumerably infinite number of coordinates showing up in the case of an anonymous description of the field of a system, is bypassed with the aid of the cell-method. In this method, the mean value of the field's amplitude in a given cell is taken as a coordinate according to the Heisenberg-Pauli assumption [4]. The original set of a denumerably infinite number of coordinates is replaced by the set of enumerably infinite number of coordinates thus obtained.

0040–5744/78/0047/0147/$02.00

The subject matter of our treatment is an interacting many-particle system in the gaseous or the fluid phase, given in an individual description. Starting from this individual description, we pass over to the anonymous field description with the aid of the hydrodynamical model.

In this paper, the velocity field appearing in the hydrodynamical model is assumed to be irrotational at first. Such a field can be represented by a scalar potential field. A velocity field representable by a vectorial potential field will be the subject of a forthcoming paper.

The determining equations for the correction fields of the different orders of perturbation are obtained with the aid of Hamilton's principle. The determining equations of different orders of approximation are in fact field equations for the respective correction field quantities. First each correction field turns out to be the solution of a characteristic field equation corresponding to the respective order of approximation. Evaluating the correction field quantities applying the classical perturbation theory we shall not be confronted with the undefinable operators which show up with the quantized hydrodynamical description [5]. It is advisable to introduce the canonically conjugated field quantity of the scalar field according to the well known definition. The expansion of the canonically conjugated field according to small powers of parameters and subsequent comparison of the coefficients of equal powers yields the framework for the quantization of the field. Finally, the quantization of the fields is achieved by representing the correction field quantities determined perturbation theoretically in terms of operators of the free (unperturbed) field.

As a special problem, we shall treat irrotational fluid helium He^4 according to our theory, showing the difference between the conventional method and ours. We shall also discuss the quantitative improvement which our theory yields when calculating the self energy.

2. Perturbation Theoretical Treatment of a Classical Field

We consider a scalar field $\varphi(x)$, originating from an interacting many-particle system. Furthermore, we shall restrict ourselves to a single kind of particles. An individual description of an interacting many-particle system leads us, with the aid of the hydrodynamical model, to an anonymous field description. Using the cell-method, the Lagrange function of a denumerably infinite number of co-ordinates can be written as a sum over all the cells:

$$L=\sum_s \tilde{L}_s\, \delta x_s^3. \tag{1}$$

A subsequent limit process leads to the continuum, and yields the Lagrange density depending on the amplitude functions of the field $\varphi(x)$ in all points of the Minkowski-space, and their spacial and time derivatives

$$\varphi_{,\mu} \equiv \frac{\partial \varphi}{\partial x_\mu}, \qquad x_\mu = \{x_1, x_2, x_3, x_4 = ict\}$$

in the following way:

$$L=\int \mathrm{d}\tau\ \tilde{L}(\varphi(x),\ \varphi_{,\mu}(x)). \tag{2}$$

The Lagrange density of the system should depend on a parameter $\sigma([0, 1])$. Let $\varphi(x;\sigma)$ be the solution to the Hamiltonian extremum problem

$$\delta A=\frac{1}{ic}\,\delta\int_1^2 \tilde{L}(\varphi,\ \varphi_{,\mu})\ \mathrm{d}^4x=0. \tag{3}$$

The numbers 1 and 2 appearing on the integral are supposed to indicate that at all the points the values of the function $\varphi(x;\sigma)$ to be varied are fixed on the border of space region at two times.

It follows from this condition that

$$\varphi^0(x)\equiv\varphi(x;\sigma=0) \tag{4}$$

is the solution to the extremum problem

$$\delta A_0=\frac{1}{ic}\,\delta\int_1^2 \tilde{L}_0(\varphi^0,\ \varphi^0_{,\mu})\ \mathrm{d}^4x=0 \tag{5}$$

where $\tilde{L}_0$ means the Lagrange density for $N=0$. It corresponds to the field quantity resulting from the non-interacting particles.

Now let us assume that the field quantity $\varphi(x;\sigma)$ at the site $\sigma=0$ can be expanded according to powers of σ

$$\varphi(x;\sigma)=\varphi^0(x)+\sigma\varphi^1(x)+\sigma^2\varphi^2(x)+\cdots. \tag{6}$$

As $\varphi(x;\sigma)$ will satisfy, for all values of σ, the boundary conditions denoted by 1 and 2 and as also $\varphi^0(x)$, being the solution to the extremum problem (5), satisfies these conditions, the correction functions $\varphi^m(x)\,(m\geqslant 1)$ have the property

$$\varphi^m(x(1))=\varphi^m(x(2))=0. \tag{7}$$

For the following we shall assume the extremum problem (5) has been solved so that the field quantity $\varphi^0(x)$ is known. We shall now solve the extremum problem (3) or, in other words, calculate the quantity $\varphi^m(x)$; $m=1, 2, \ldots$. To achieve this, we shall expand the Lagrangian density $\tilde{L}$ into a Taylor series $\tilde{L}=\sum_{l=0}^{\infty}\sigma^l\tilde{L}_l$. To derive the Lagrangian density $\tilde{L}_l$ according to the parameter σ, we avail ourselves of the rules for implicit functions. According to this method, the Lagrangian density $\tilde{L}$ and the derivatives of $\tilde{L}_k$ at the site $\sigma=0$, which we shall designate by

$$G_k^{\alpha,\beta}=\left(\frac{\partial^{\alpha+\beta}\tilde{L}_k}{\partial\varphi^\alpha\,\partial\varphi^\beta_{,\mu}}\right)_{\sigma=0} \tag{8}$$

can be written as

$$\tilde{L}=\sum_{l} \sigma^l \Lambda_l$$

$$\Lambda_l=\sum_{k=0}^{l} \frac{(1+\delta_{\alpha\beta})}{\prod_{\lambda}(\lambda\alpha_\lambda+\lambda\beta_\lambda)!} G_k^{\alpha,\beta}(\varphi^\lambda)^{\alpha_\lambda}(\varphi^\lambda_{,\mu})^{\beta_\lambda} \tag{9}$$

where

$$(\varphi^\lambda)^{\alpha_\lambda}=(\varphi^1)^{\alpha_1}(\varphi^2)^{\alpha_2}\ldots, \qquad (\varphi^\lambda_{,\mu})^{\beta_\lambda}=(\varphi^1_{,\mu})^{\beta_1}(\varphi^2_{,\mu})^{\beta_2}\ldots \tag{10}$$

$$\alpha=\sum_\lambda \alpha_\lambda, \qquad \beta=\sum_\lambda \beta_\lambda,$$

$$\lambda\alpha_\lambda+\lambda\beta_\lambda=1\alpha_1+2\alpha_2+\cdots+1\beta_1+2\beta_2+\cdots \quad \text{and} \quad \alpha+\beta+k=l.$$

Inserting (9) into (3), the variational problem gets the new shape

$$\delta A=\frac{1}{ic}\sum_{l=0}^{\infty} \sigma^l\, \delta \int_1^2 \Lambda_l\, d^4x=0. \tag{11}$$

Now, the extremum problem (11) contains a set of functions $\varphi^m(x)$ and $\varphi^m_{,\mu}(x)$ to be varied. At the boundaries 1 and 2, all functions $\varphi^m(x)$ assume the value 0. These are the conditions for the functions $\varphi^m(x)$ which have to be considered when the variational problem is solved.

The field equations equivalent to (11) are

$$\sigma^l\left[\frac{\partial}{\partial\varphi^m}-\left(\frac{\partial}{\partial\varphi^m_{,\mu}}\right)_{,\mu}\right]\Lambda_l=0. \tag{12}$$

The determining equation for the correction field quantity of order m is obtained by setting all the powers of σ equal to 0 as the functions $\varphi^m(x)$ do not depend on the parameter σ. The determining equations for the different orders of correction have to be considered as the respective field equation for the correction field. The first equation thus obtained (for $l=1$, $k=0$ and α, β) satisfying conditions (10), i.e.

$$G_0^{1,0}-G_{,\mu}^{0,1}=0 \tag{13}$$

does not yield anything of novelty character. It only shows that the field quantity $\varphi^0(x)$ is the solution to the field equation

$$\frac{\partial\tilde{L}_0}{\partial\varphi^0}-\left(\frac{\partial\tilde{L}_0}{\partial\varphi^0_{,\mu}}\right)_{,\mu}=0 \tag{14}$$

of the extremum problem (3).

However, the equation for $l=2$, $k=0$, and $l=2$, $k=1$ and the suitable α and β values is the determining equation for the correction field quantity of the first order $\varphi^1(x)$:

$$\left[G_0^{0,2}\frac{\partial^2}{\partial x_\mu^2}+G_{0,\mu}^{0,2}\frac{\partial}{\partial x_\mu}+(G_{0,\mu}^{1,1}-G_0^{2,0})\right]\varphi^1(x)=-(G_{1,\mu}^{0,1}-G_1^{1,0}). \tag{15}$$

For the special case of one degree of freedom, Eq. (15) turns out to be the determining equation for q_1 [1].

To get a handier expression for the field equation of higher correction order, we shall designate the differential operators with the aid of the symbol

$$\Omega_k \equiv G_k^{0,2} \frac{\partial^2}{\partial x_\mu^2} + G_{k,\mu}^{0,2} \frac{\partial}{\partial x_\mu} + (G_{k,\mu}^{1,1} - G_k^{2,0}) \quad (k=0, 1, 2, \ldots) \tag{16}$$

and the terms of the inhomogeneity by the symbol

$$\rho_k(x) = G_{k+1,\mu}^{0,1} - G_{k+1}^{1,0} \quad (k=0, 1, 2, \ldots). \tag{17}$$

The coefficients of the powers σ^2 and σ^3 supply respectively the field equation for the unperturbed field and the correction field of first order. The coefficients of the powers σ^4 yield the field equation for the correction of second order:

$$\Omega_0 \varphi^2(x) = -\rho_1(x) - \Omega_1 \varphi^1(x). \tag{18}$$

The field equation for the correction of order m is obtained from the coefficient of σ^{2m}:

$$\Omega_0 \varphi^m(x) = -\rho_{m-1}(x) - \sum_{k=1}^{m-1} \Omega_k \varphi^{m-k} \quad (m \geqslant 2) \tag{19}$$

which, in the inhomogeneous term, only contains the functions $\varphi^n(x)(n<m)$ which are known.

It should be stressed that the theory, owing to the expansion according to the small parameter σ, supplies surplus information for the Lagrangian density as well as for the field quantity. For this reason, the determining equations for corrections of higher order are obtained from the coefficients of σ^{2m}, $m=1, 2, \ldots$. However, the theory opens up the possibility to consider the coupling between the unperturbed and perturbed fields; for $l=1$, e.g., the theory contains apart from the Lagrangian density of the conventional quantum field theory also the coupling terms:

$$\Lambda_1 = (\tilde{L}_1)_{\sigma=0} + \left(\frac{\partial \tilde{L}_0}{\partial \varphi}\right)_0 \varphi^1 + \left(\frac{\partial \tilde{L}_0}{\partial \varphi_{,\mu}}\right)_0 \varphi^1_{,\mu}.$$

Field equation (19) for the correction of order m is a linear differential equation of second order and is therefore easy to handle. It is soluble in case the correction fields of orders lower than that of $\varphi^m(x)$ are known; but it is by no means the same field equation as the one satisfied by the unperturbed field.

It seems expedient, for the quantization of the field to be carried out later, to introduce the canonically conjugated field quantity at $\eta(x)$ of the field $\varphi(x)$ which, according to the definition, will be $\eta(x) = \partial\tilde{L}/\partial\dot{\varphi}$. Here we expand according to the small parameter

$$\begin{aligned} \eta(x) &= \eta^0(x) + \sigma^1 \eta^1(x) + \sigma^2 \eta^2(x) + \cdots \\ &= \sum_l \sum_j \sigma^l \left(1 + \sigma^j \frac{\partial \dot{\varphi}^1}{\partial \dot{\varphi}^0}\right)^{-1} \frac{\partial \tilde{L}_l}{\partial \dot{\varphi}^0}. \end{aligned} \tag{20}$$

By a comparison of the coefficients, we successively get the canonically conjugated field quantities $\eta^0(x)$ and their correction quantities:

$$\eta^0(x) = \frac{\partial \tilde{L}_0}{\partial \dot{\varphi}^0}$$

$$\eta^1(x) = \frac{\partial \tilde{L}_1}{\partial \dot{\varphi}^0} - \eta^0(x)\frac{\partial \dot{\varphi}^1}{\partial \dot{\varphi}^0} \tag{21}$$

$$\eta^2(x) = \frac{\partial \tilde{L}_2}{\partial \dot{\varphi}^0} - \frac{\partial \tilde{L}_1}{\partial \dot{\varphi}^0}\frac{\partial \dot{\varphi}^1}{\partial \dot{\varphi}^0},$$

$$\vdots$$

to be used for the quantization of the fields.

3. The Quantization of the Fields

The quantization of the non-interacting fields is carried out as usual, i.e. by requiring that the canonical field quantities φ^0 and η^0 satisfy the commutation relation

$$[\eta^0(x), \varphi^0(x')] = \frac{\hbar}{i}\,\delta(x - x'). \tag{22}$$

As the correction quantities have already been determined by the above classical treatment, we only need to represent them by operators.

The determining equation for the classical correction field of first order now has to be transformed into a quantum theoretical operator equation the formal solution of which will have the following shape owing to the rules of the operator calculus:

$$\varphi^1(x) = -\Omega_0^{-1}\rho_0(x). \tag{23}$$

In this equation, Ω_0^{-1} designs the operator reciprocal to Ω_0. If $\rho_0(x)$ has a definite value at x it can be written with the aid of $\delta(x)$ function as follows:

$$\rho_0(x) = \int d^4x'\,\rho_0(x')\,\delta(x - x'). \tag{24}$$

If we substitute this into (23) taking into account that the reciprocal differential operator Ω_0^{-1} only acts upon the unprimed variables, the solution of the operator equation equivalent to the field equation for the correction of first order gets the form

$$\varphi^1(x) = \int d^4x'\,\rho_0(x')(-\Omega_0^{-1}\,\delta(x - x')). \tag{25}$$

Introducing Green's function $G(x, x')$, which is defined by

$$G(x, x') = -\Omega_0^{-1}\,\delta(x - x')$$

we finally get:

$$\varphi^1(x) = \int d^4x' G(x, x')\, \rho_0(x'). \tag{26}$$

By the same method, the correction of second order is obtained in the form

$$\varphi^2(x) = \int d^4x'' G(x, x'')\rho_1(x'') + \int\int d^4x''\, d^4x' G(x, x'')\Omega_1 G(x'', x')\rho_0(x'). \tag{27}$$

By repeating this procedure, corrections of arbitrarily high order containing only known operators can be obtained. Having the operator representations for the correction fields $\varphi^m(x)$, also the canonically conjugated fields $\eta^m(x)$ can easily be written in terms of operators.

4. Application of the Theory to the Phonodispersion in He^4

The Lagrangian density of the irrotatory fluid helium can be written, according to the hydrodynamical model [6], as a function of the velocity potential $v = \nabla\varphi$, the density μ and the density fluctuation $\eta = \mu - \mu_0$, as

$$\tilde{L} = \eta\, \dot{\varphi} - \frac{\mu}{2} v^2 - \mu\varepsilon(\mu). \tag{28}$$

In this equation, $\varepsilon(\mu)$ is the internal energy per unit mass of the fluid. The expansion of the Lagrangian density according to ascending powers of the density fluctuation η leads to the coefficient [7]

$$\begin{aligned}
\tilde{L}_0 &= \eta\, \dot{\varphi} - \frac{\mu_0}{2}(\nabla\varphi)^2 - \frac{c^2}{2\mu_0}\eta^2 \\
\tilde{L}_1 &= -\tfrac{1}{2}\nabla\varphi\eta\nabla\varphi - \frac{1}{3!}\left(\frac{d}{d\mu}\frac{c^2}{\mu}\right)_0 \eta^3 \\
\tilde{L}_2 &= -\frac{1}{4!}\left(\frac{d^2}{d\mu^2}\frac{c^2}{\mu}\right)_0 \eta^4.
\end{aligned} \tag{29}$$

The quantization of the free field can be carried out in the usual manner, and the quanta turn out to be the elementary excitations of helium in the lower range of impulse, i.e. the phonons [8].

The differential operator Ω_0 and the terms of inhomogeneity ρ_0 used for the operator representation of the correction field φ^1 have, for the Lagrangian densities $\tilde{L}_0$ and $\tilde{L}_1$ in (29), the following shape:

$$\Omega_0 = -\mu_0 \Delta \tag{30}$$

and

$$\rho_0(x) = -\nabla\tfrac{1}{2}(\eta^0\nabla\varphi^0 + \nabla\varphi^0\eta^0) \equiv -\nabla\{\eta^0\nabla\varphi^0\}_+. \tag{31}$$

According to formula (25), the correction of first order is given by

$$\varphi^1 = \frac{1}{\mu_0} \int d^3x' \nabla' \{\eta^0 \nabla' \varphi^0\}_+ (-\Delta^{-1}\, \delta(\boldsymbol{x}-\boldsymbol{x}')). \tag{32}$$

The primes at the Nabla-operators are to show that the derivation has been carried out according to the primed coordinates. The Green's function defined by $G(\boldsymbol{x}, \boldsymbol{x}') = -\Delta^{-1}\, \delta(\boldsymbol{x}-\boldsymbol{x}')$ has the solution

$$G(\boldsymbol{x}-\boldsymbol{x}') = (4\pi|\boldsymbol{x}-\boldsymbol{x}'|)^{-1}\, e^{ik|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{33}$$

The operators appearing in the integrand of (32) can be written, using the Fourier-representation of φ^0 and η^0 as

$$\nabla'\{\eta^0, \nabla'\varphi^0\}_+ = \frac{1}{V} \sum_{\boldsymbol{k},\boldsymbol{k}'} (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{k}'^2)\{p_{\boldsymbol{k}} q_{\boldsymbol{k}'}\}_+\, e^{-i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}'}. \tag{34}$$

Putting in (33) and (34) into (32) and carrying out some integral calculation, the operator representation of the correction of first order gets the following shape:

$$\varphi^1 = \frac{1}{\mu_0 V} \sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{k}'}{|\boldsymbol{k}-\boldsymbol{k}'|^2 - k''^2} \{p_{\boldsymbol{k}} q_{\boldsymbol{k}'}\}_+\, e^{-i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}'}. \tag{35}$$

The operator φ^1 is identical to zero in case both states $\boldsymbol{k}$ and $\boldsymbol{k}'$ are identical. As the interaction between the two fields is due to mutual perturbations of wave fields of different wave lengths, the vanishing of operator φ^1 for $\boldsymbol{k}=\boldsymbol{k}'$ is understandable from the physical standpoint. The expression for φ^1 in (35) can be represented with the aid of the phonon creation and phonon annihilation operators as

$$\varphi^1 = \frac{\hbar}{2\mu_0 V} \sum_{\boldsymbol{k},\boldsymbol{k}'} i\sqrt{\frac{k}{k'}} \frac{(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{k}'}{|\boldsymbol{k}-\boldsymbol{k}'|^2 - k''^2} \{(a_{\boldsymbol{k}}^* - a_{\boldsymbol{k}})(a_{\boldsymbol{k}'} + a_{-\boldsymbol{k}'}^*)\}_+ \exp\{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}'\}. \tag{36}$$

Thus it can be seen that the correction field concerns two quantum processes. The canonically conjugated field η^1 corresponding to φ^1 can be expressed with the aid of formulae (21) and (36) as

$$\eta^1 = \frac{\hbar}{2Vc} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{l}} \frac{(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{k}'}{|\boldsymbol{k}-\boldsymbol{k}'|^2 - k''^2} \sqrt{k\cdot l}\, (a_{\boldsymbol{l}}^* - a_{\boldsymbol{l}})(a_{\boldsymbol{k}}^* - a_{-\boldsymbol{k}}) \exp\{-i(\boldsymbol{l}+\boldsymbol{k})\boldsymbol{x}'\}. \tag{37}$$

From the expansion of the Hamiltonian density according to the small parameter σ, one gets

$$\tilde{H} = (\tilde{H}_0)_{\sigma=0} + \sigma\left[(\tilde{H}_1)_{\sigma=0} + \left(\frac{\partial \tilde{H}_0}{\partial \nabla\varphi}\right)_{\sigma=0} \nabla\varphi^1 + \left(\frac{\partial \tilde{H}_0}{\partial \eta}\right)_{\sigma=0} \eta^1\right] + \cdots. \tag{38}$$

Apart from $(\tilde{H}_1)_{\sigma=0}$, the interaction terms originate from $(\tilde{H}_1)_0$ and also from the other terms which represent the coupling between free and perturbed fields. The additional terms can be expressed with the aid of formulae (36) and (37) in the following manner:

$$\left(\frac{\partial \tilde{H}_0}{\partial \nabla\varphi}\right)_{\sigma=0} \nabla\varphi^1 + \left(\frac{\partial \tilde{H}_0}{\partial \eta}\right)_{\sigma=0} \eta^1 = \frac{1}{V} \sum_{l,m,k} \frac{(l-m)^2}{|l-m|^2 - k^2} \nabla\varphi^0 \eta^0 \nabla\varphi^0 - \frac{c^2}{\mu_0 V} \sum_{l,m,k} \frac{(l-m).m}{|l-m|^2 - k^2} \eta^{0^3}. \tag{39}$$

As the order of the additional terms is the same as that of

$$(\tilde{H}_1)_{\sigma=0} = \tfrac{1}{2} \nabla\varphi^0 \eta^0 \nabla\varphi^0 + \frac{1}{3!}\left(\frac{d}{d\mu}\frac{c^2}{\mu}\right)_0 \eta^{0^3} \tag{40}$$

the total interaction can be summarized in the following way [9]:

$$\tilde{H}_{\text{int}} = \sum_{k,l,m} V(k,l,m)\{\tfrac{1}{3} a_k a_l a_m\, \delta(k+l+m) + a_k a_l^* a_m^*\, \delta(k-l+m)\} + \text{H.C.} \tag{41}$$

the coupling constants being

$$V(k,l,m) = -i\left(\frac{\hbar^3 c}{32\mu_0 V}\right)^{1/2} \sqrt{klm} \cdot \{\hat{k}\cdot\hat{l} f_k(l,m) + \hat{l}\cdot\hat{m} f_l(k,m) + \hat{k}\cdot\hat{m} f_k(l,m) + (2u-1)g(u)\}. \tag{42}$$

Here $\hat{k} = k/k$, $u = \mu_0/c(\partial u/\partial\mu)_0$ and the numerical factors are

$$f_k(l,m) = 1 - 2\frac{(l-m)^2}{|l-m|^2 - k^2}, \qquad g(u) = 1 - \frac{(l-m),m}{|l-m|^2 - k^2}(2u-1)^{-1}. \tag{43}$$

This theory differs from the conventional theory only by the coupling constants. For $f_k(l,m)=1$ and $g(u)=1$ the two theories correspond to each other.

The energy perturbation $\Delta\varepsilon_p$ at 0°K can be determined, using the thermodynamical perturbation theory as in Ref. [10], by calculating all the contributions to self energy of lowest order and by letting $T \to 0$, as Eckstein and Varga have shown [11]. Assuming that the total number of longitudinal phonons corresponds to the density of the number of helium atoms, the cut off impulse K can be estimated by $K/\hbar = (6\pi^2 N)^{1/3}$. With this value K, the energy perturbation is obtained.

$$\Delta\varepsilon_p = -\frac{K^4}{32\pi^2 \mu_0 \hbar^3} p\{(u-1)^2 g^2(u) + \tfrac{2}{3}(p/K)^2 (u^2 g^2(u) - \tfrac{3}{5} f^2)\}. \tag{44}$$

After a sort of renormalization of the velocity of sound [11], $c' = c - \Delta c$ with $\Delta c = K^4 (u-1)^2 g^2(u)/(32\pi^2 \mu_0 \hbar^3)$, one gets, as expected, the Landau phonon energy dispersion

$$\varepsilon = \varepsilon_p + \Delta\varepsilon_p = c'p(1 - \gamma p^2) \tag{45}$$

where $\gamma = (K^2/48\pi^2 c' \mu_0 \hbar^3)(u^2 g^2(u) - 0.6 f^2)$ with the data $K/\hbar = 1.09$ Å^{-1} and $u = 2.6 \times 10^{37}$ cgs units we get $\gamma = 2.27 \times 10^{37}$ cgs units. This agrees well with the value $\gamma = 2.8 \times 10^{37}$ cgs units estimated by neutron scattering [12]. The conventional theory [11] yields a value of $\gamma = 4.3 \times 10^{37}$ cgs units.

5. Discussion

By the extension of classical perturbation theory which has been developed on the basis of Hamilton's principle, a many-particle problem has been treated perturbation theoretically within the scope of a hydrodynamic field. Corrections of arbitrarily high orders can be obtained with the aid of determining equations for the respective orders, which turn out to be linear differential equations of second order. This means that correction fields of higher orders can be calculated iteratively once the unperturbed problem has been solved. In this work, we have at first treated the case in which the field of velocity of fluid can be represented by a scalar potential field. The corrections calculated perturbation theoretically were quantized and can then be represented by known operators. This theory was then applied to the problem of phonon dispersion in fluid helium He^4. We found that this theory differs from the conventional as it also accounts for the coupling between free and perturbed fields in addition to conventional interaction. These contributions can be combined with the conventional interaction term as they depend in the same order on the operators as the conventional one does. Therefore, this new theory differs from the conventional by the coupling constants. The calculation for the γ value yields a very good agreement with experimental values. It should therefore be stressed that the hydrodynamical model is a useful basis for the description of an interacting many-particle system.

Acknowledgements. One of us (K.M.C.) wishes to thank the Akademie der Wissenschaften und der Literatur zu Mainz for financial support. The authors furthermore wish to thank the editorial office of *Theoretica Chimica Acta* for translational help.

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Received September 14, 1977